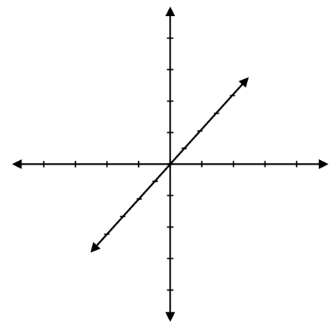
SECTION 13.6: SURFACES

FREE VARIABLES AND 'CYLINDERS'

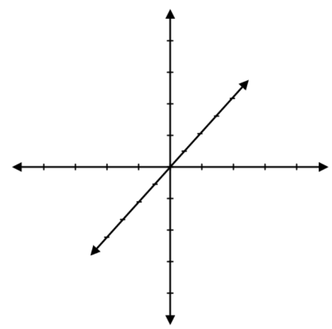
RECALL: The **graph** of an equation is the set of all points which make the equation true.

EXAMPLE 1: Sketch or otherwise describe the graph of the given equation.

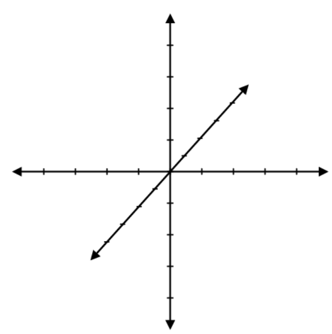
1. $x + 2y = 6$



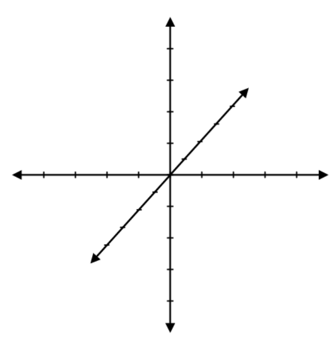
2. $z = x^2$



3. $x^2 + y^2 = 9$



4. $z = \sin(\pi y)$



SURFACES OF REVOLUTION

RECALL: A **surface of revolution** is obtained by rotating a curve in the xy- plane about either the x- or y-axis. This results in a surface whose cross sections are circles centered on the axis of rotation.

EQUATIONS OF SURFACES OF REVOLUTION:

- Rotating $y = f(x)$ about the x-axis: $y^2 + z^2 = [f(x)]^2$
- Rotating $x = f(y)$ about the y-axis: $x^2 + z^2 = [f(y)]^2$

EXAMPLE 2:

1. Write the equation of the surface obtained by revolving $y = \sin(x)$ about the x-axis.

Ans: $y^2 + z^2 = \sin^2(x)$

2. Describe the graph of $y^2 + z^2 = x$ by identifying it as a surface of revolution.

Ans: revolved $y = \sqrt{x}$ about the x-axis.

3. Describe the graph of $x^2 + z^2 = y^2$ by identifying it as a surface of revolution.

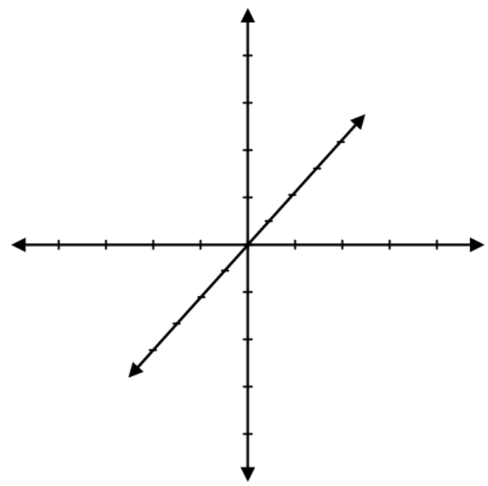
Ans: revolved $x = y$ about the y-axis.

TRACES, SLICES, CONTOUR MAPS

DEFINITION: The **trace** of a surface in a coordinate plane is the curve of intersection of the surface and the coordinate plane. More specifically:

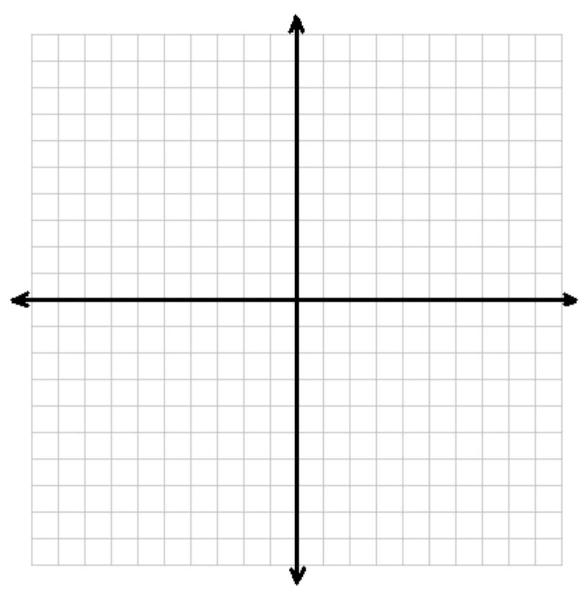
- The xy-trace is the intersection of the surface and the plane $z = 0$ (the xy-plane.)
- The yz-trace is the intersection of the surface and the plane $x = 0$ (the yz-plane.)
- The xz-trace is the intersection of the surface and the plane $y = 0$ (the xz-plane.)

EXAMPLE 3: Find the traces of $x + 2y + 3z = 6$. Sketch the portion of this plane which lies in the first octant.



More generally, we can **slice** a surface with planes parallel to the coordinate planes to help get a sense of what the surface looks like. A '**contour map**' of a surface is found by sketching several representative '**contours**' or '**level curves**' of the surface obtained by slicing the surface with horizontal planes of the form $z = c$ in the xy-plane.

EXAMPLE 4: Make a contour map for $z = x^2 + y^2$.



REVIEW OF THE CONIC SECTIONS

GEOMETRIC DEFINITION: 'Conic sections' come from slicing the cone at different angles:

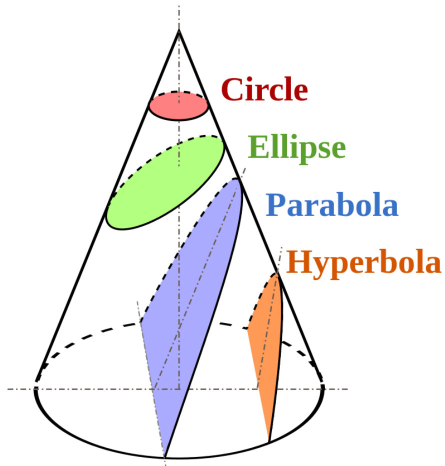
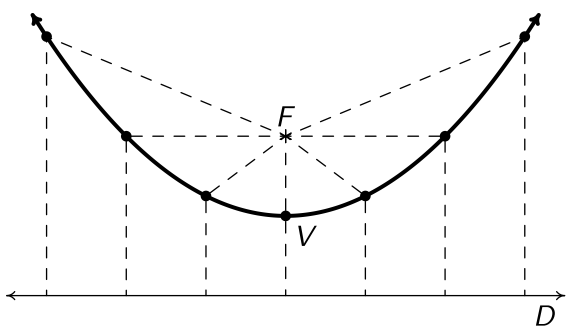


Image taken from Wikipedia.

They can also be described as a **locus** (set) of points in the plane which satisfy certain distance conditions.
ALGEBRAIC DEFINITION: The conic sections arise from graphing quadratic equations in two variables:

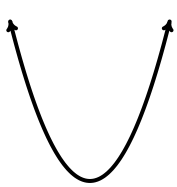
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

PARABOLAS: Given a fixed line called the **directrix**, D, and a fixed point not on the line called the **focus**, F, a **parabola** is the set of all points whose distance to the directrix is the same as their distance to the focus.

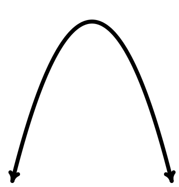


Equations of parabolas have only **one** squared term, and there are two standard forms:

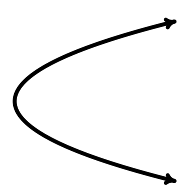
$(x-h)^2 = 4p(y-k)$ $\qquad\qquad$ $(y-k)^2 = 4p(x-h)$



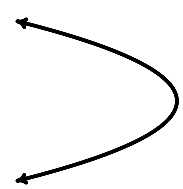
$p > 0$



$p < 0$



$p > 0$

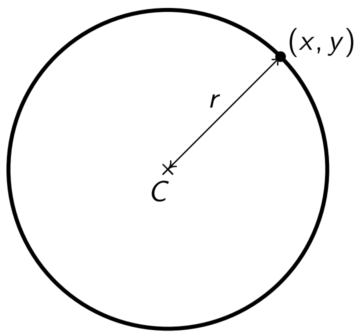


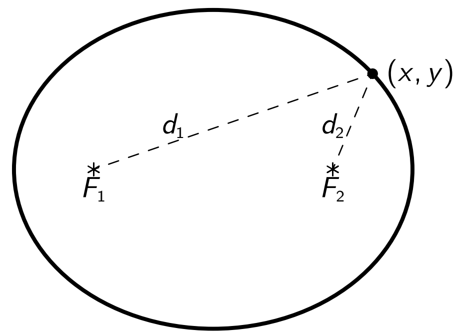
$p < 0$

In each case:

- the vertex is (h, k).
- p is the *directed* distance from the vertex to the focus (and from the vertex to the directrix.)
- $|4p|$ is the width of the parabola at the focus (the length of the latus rectum)

CIRCLES AND ELLIPSES: Given a fixed point called the **center**, C, and a fixed distance called the **radius**, r, a **circle** is the set of all points r units away from the center. Given two fixed points called **foci**, F_1 and F_2, and a fixed distance d, an **ellipse** is the set of all points the **sum** of whose distances to the foci is d.

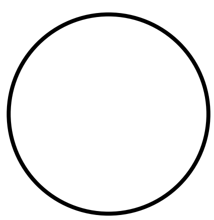




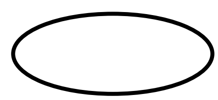
$d_1 + d_2 = d$ for all (x, y) on the ellipse

Equations of circles and ellipses contain **two** squared terms with the **same** signs. Equal coefficients point to a circle; unequal coefficients indicate an ellipse. The standard form is:

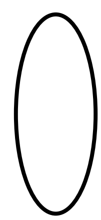
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



$a = b$



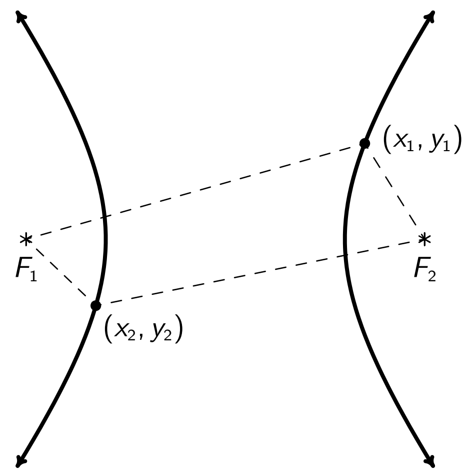
$a > b$



$b > a$

- The center in each case is (h, k).
- 'a' tells us how far to move in x-direction from the center (i.e., the horizontal stretch.)
- 'b' tells us how far to move in the y-direction from the center. (i.e., the vertical stretch.)
- If $a = b$, the curve is a circle, and $a = b = r$ is the radius of the circle.
- If $a > b$, the curve is an ellipse with a horizontal major axis.
  - The vertices are 'a' units to the left and right of the center.
  - The foci are $c = \sqrt{a^2 - b^2}$ units to the left and right of the center.
- If $b > a$, the curve is an ellipse with a vertical major axis.
  - The vertices are 'b' units above and below the center.
  - The foci are $c = \sqrt{b^2 - a^2}$ units above and below the center.
- The eccentricity, $e = \dfrac{\text{distance from center to focus}}{\text{distance from center to vertex}} = \dfrac{c}{a \text{ or } b\text{, whichever is bigger}}$
  - e is a measure of 'roundness' and for ellipses, $0 < e < 1$.
  - If $e \approx 0$, the ellipse is more circular; if $e \approx 1$, the ellipse is less circular (more 'eccentric.')

HYPERBOLAS: Given two fixed points called **foci**, F_1 and F_2, and a fixed distance d, a **hyperbola** is the set of all points the **difference** of whose distances to the foci is d.



In the figure above:

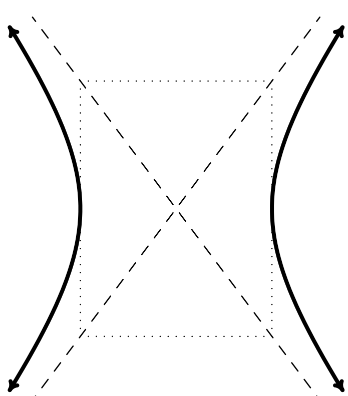
$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) = d$$

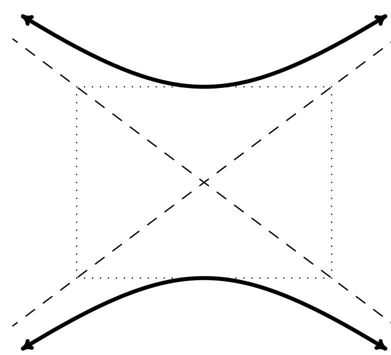
and

$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) = d$$

Equations of hyperbolas contain **two** squared terms with **different** signs. There are two standard forms:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad\qquad \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$





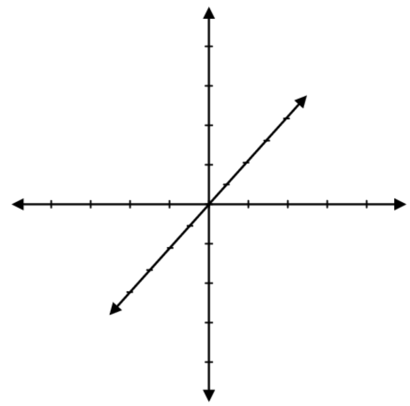
- The center in each case is (h, k).
- To sketch a 'guide rectangle:' from the center, move 'a' units left and right and 'b' units up and down.
- The slant asymptotes contain contain the center, (h, k) and have slopes $\pm b/a$.
- If the standard form contains $x^2 - y^2$, the curve is a hyperbola with a horizontal transverse axis.
  - The vertices are 'a' units to the left and right of the center.
  - The foci are $c = \sqrt{a^2 + b^2}$ units to the left and right of the center.
- If the standard form contains $y^2 - x^2$, the curve is a hyperbola with a vertical transverse axis.
  - The vertices are 'b' units above and below the center.
  - The foci are $c = \sqrt{b^2 + a^2}$ units above and below the center.

THE QUADRIC SURFACES

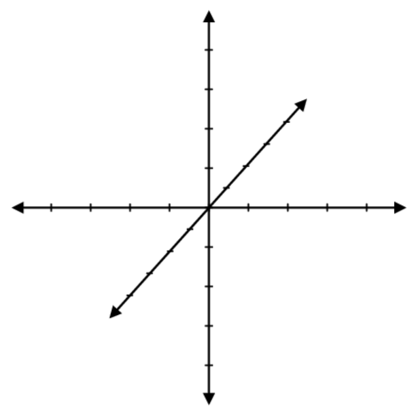
A **quadric surface** is the graph of a second degree equation in **three** variables, much like the **conic sections** were graphs of second degree equations in **two** variables.

EXAMPLE 5: Sketch or otherwise describe the graphs of the following equations using traces and slices.

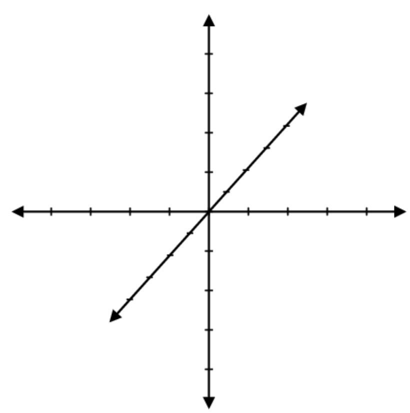
1. $9x^2 + 9y^2 + 4z^2 = 36$



2. $x^2 + y^2 - z^2 = 4$



3. $z = x^2 - y^2$



HOMEWORK: Section 13.6: 7 - 67 every other odd; 64*